ปริภูมิอิงระยะทางแบบ $b-$มุมฉากค่าเชิงซ้อน และทฤษฎีบทจุดตรึง

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Complex valued rectangular $b-$metric spaces and fixed point theorems

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ชื่อโครงการวิจัย	ปริภูมิอิงระยะทางแบบ $b-$มุมฉากค่าเชิงซ้อนและทฤษฎีบทจุดตรึง
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Abstract

ในงานวิจัยนี้ เราศึกษาการมีอยู่ของจุดตรึงสำหรับการส่งด้วยตนเองภายใต้แนวคิดการส่งแบบทั่วไปของ Kannan ในปริภูมิอิงระยะทางแบบ b มุมฉาก ผลลัพธ์ของเราจะขยายออก และสรุปผลลัพธ์ที่ได้จาก Ege [21] และอื่น ๆ อีกมากมาย

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Abstract

In this paper, we study the existence of fixed point for self mappings under generalized kannan mappings type concept in rectangular $b-$metric spaces. Our result extend and generalize the result derived by Ege [21] and many others.

คำนำ

รายงานผลการวิจัย เรื่อง ปริภูมิอิงระยะทางแบบ ๖-มุมฉากค่าเชิงซ้อนและทฤษฎีบทจุดตรึง ซึ่งจัดทำโดย นางสาวมุธิตา ชะดาจันทร์ และอาจารย์ที่ปรึกษา ดร.ชลธิศ เสือนุ่ม สังกัดโปรแกรมวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏกำแพงเพชร โครงการศึกษานี้ได้รับเงินสนับสนุนจาก สถาบันวิจัยและพัฒนา มหาวิทยาลัยราชภัฏกำแพงเพชร โดยมีวัตถุประสงค์ดังต่อไปนี้ (๑) เราจะศึกษาการมีอยู่ของจุดตรึงสำหรับการส่งด้วยตัวเองภายใต้แนวคิดของการส่งแบบคานโดยทั่วไปในปริภูมิอิงระยะทางแบบ ๖-มุมฉากค่าเชิงซ้อน (๒) พิสูจน์ทฤษฎีบทในปริภูมิอิงระยะทางแบบ ๖-มุมฉากค่าเชิงซ้อน

ผู้วิจัยหวังเป็นอย่างยิ่งว่า รายงานผลการวิจัยฉบับนี้จะเป็นประโยชน์ทางด้านวิชาการ และการต่อยอดการศึกษาและวิจัยในระดับสูง อันจะเป็นพื้นฐานในการพัฒนาประเทศชาติต่อไป

คณะผู้วิจัย
๒๒ กุมภาพันธ์ ๒๕๖๒

กิตติกรรมประกาศ

ข้าพเจ้าขอขอบคุณสถาบันวิจัยและพัฒนา มหาวิทยาลัยราชภัฏกำแพงเพชร ที่ให้ทุนสนับสนุนสำหรับการศึกษาวิจัย

รายงานการศึกษาวิจัยนี้ สำเร็จลุล่วงได้ด้วยความกรุณาช่วยเหลือ แนะนำ ให้คำปรึกษา ตรวจสอบแก้ไขข้อบกพร่องต่าง ๆ ด้วยความเอาใจใส่อย่างดียิ่งจาก ดร.ชลธิศ เสือนุ่ม จากโปรแกรมวิชาคณิตศาสตร์ คณะวิทยาศาสตร์และเทคโนโลยี มหาวิทยาลัยราชภัฏกำแพงเพชร ผู้เขียนกราบขอบพระคุณเป็นอย่างสูง

ขอขอบคุณ อาจารย์โปรแกรมวิชาคณิตศาสตร์ มหาวิทยาลัยราชภัฏกำแพงเพชร ที่ช่วยสนับสนุนการทำวิจัยในทุก ๆ ด้าน และเป็นกำลังใจตลอดมา

ขอขอบคุณ ผู้ที่ให้ความร่วมมือช่วยเหลือทุกท่าน และญาติพี่น้องทุกคนที่ช่วยเหลือสนับสนุนทั้งด้านกำลังใจ ด้วยดีตลอดมา จึงขอขอบคุณทุกท่านเหล่านั้นไว้ ณ โอกาสนี้ด้วย

องค์ความรู้ และคุณค่าทั้งหลายที่ได้รับจากรายงานการวิจัยฉบับนี้ ผู้เขียนขอมอบเป็นกตัญญูกตเวทีแด่บิดามารดา และบูรพาจารย์ที่เคยอบรมส่งสอน รวมทั้งผู้มีพระคุณทุกท่าน

มุธิตา ชะดาจันทร์
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Content

Chapter 1

Introduction

The Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. Metric fixed point theory the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space was introduced by Frechet [1]. In 2000, Branciari [2] introduced a notion of rectangular metric space and proved an analogue of the Banach contraction principle in this space, then various fixed point theorems were given for different contractions on rectangular metric spaces (see [3]- [11]). In 2011, Azam and et al.[12] introduced the complex valued b-metric spaces and proved common fixed point results for mappings.

On the other hand, in 1989, Bakhtin[13] introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [14]- [20] and the references therein). In 2015, Ege [21] introduced complex valued rectangular b-metric spaces and proved an analogue of Banach contraction principle. Moreover, author also prove a different contraction principle with a new condition and a fixed point theorem in this space.

In this paper, we study the existence of fixed point for self mappings under generalized kannan mappings type concept in b-metric spaces. Our result extend and generalize the result derived by Ege [21] and many others.Moreover, we give examples as a satisfying the theorems in complex valued rectangular b-metric spaces.

Chapter 2

Basic knowledge

In this chapter, we review the basic knowledge to prove our main results.

Difinition 2.1. [4] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(bM1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(bM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(bM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then d is called a b-metric on X and (X, d) is called a *b-metric space (in short bMS)* with coefficient s.

Difinition 2.2. [5] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(RM1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(RM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$. Then d is called a rectangular metric space on X and (X, d) is called a *rectangular metric space (in short RMS)*.

Difinition 2.3. [22] Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(RbM1) $d(x, y) = 0$ if and only if $x = y$;

(RbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(RbM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \backslash \{x, y\}$. Then d is called a rectangular b-metric on X and (X, d) is called a *rectangular b-metric space (in short RbMS)* with coefficient s.

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse of the above implication is not necessarily true.

Example 2.4. [22] Let $X = \mathbb{N}$, define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0, & if \quad x = y; \\ 4\alpha & if \quad x, y \in \{1, 2\} \ and \ x \neq y; \\ \alpha & if \quad x \ or \ y \notin \{1, 2\} \ and \ x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space,as $d(1, 2) \ = 4\alpha > 3\alpha \ = d(1, 3) + d(3, 4) + d(4, 2)$.

Example 2.5. [22] Let $X = \mathbb{N}$, define $d : X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & if \quad x = y; \\ 10\alpha & if \ \ x = 1, \ y = 2; \\ \alpha & if \ \ x \in \{1,2\} \ and \ y \in \{3\}; \\ 2\alpha & if \ \ x, \in \{1,2,3\}, \ and \ y \in \{4\}; \\ 3\alpha & if \ \ x \ or \ y \notin \{1,2,3,4\} \ and \ x \neq y, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $s = 2 > 1$, but (X, d) is not a rectangular metric space,as $d(1, 2) \ = 10\alpha > 5\alpha \ = d(1, 3) + d(3, 4) + d(4, 2)$.

The complex metric space was initiated by Azam et al. [5]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:
(C_1) Re(z_1) = Re(z_2) and Im(z_1) = Im(z_2),
(C_2) Re(z_1) < Re(z_2) and Im(z_1) = Im(z_2),
(C_3) Re(z_1) = Re(z_2) and Im(z_1) < Im(z_2),
(C_4) Re(z_1) < Re(z_2) and Im(z_1) < Im(z_2).
Particularly, we write $z_1 \not\precsim z_2$ if $z_1 \neq z_2$ and one of (C_2),(C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:
(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $a_z \prec b_z$ for all $z \in \mathbb{C}$
(2) If $0 \precsim z_1 \not\precsim z_2$, then $|z_1| < |z_2|$.
(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Difinition 2.6. [21] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to \mathbb{C}$ satisfies:

(CRb1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(CRb2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(CRb3) there exists a real number $s \geq 1$ such that $d(x,y) \preceq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x,y\}$. Then d is called a complex valued rectangular b-metric on X and (X,d) is called a complex valued rectangular b-metric space.

Example 2.7. [21] Let X = A $\cup B$ where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ be defined as follows:

$$d(x,y) = d(y,x)$$

for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & if \quad x = y; \\ 2t, & if \;\; x, y \in A; \\ \frac{t}{2n}, & if \;\; x \in A \; and \; y \in \{2,3\}; \\ \text{t}, & otherwise, \end{cases}$$

where $t > 0$ is a constant. Then (X,d) is a complex valued rectangular b-metric space with coefficient $s = 2 > 1$.

Difinition 2.8. [21] Let (X,d) be a complex valued rectangular b-metric space, $\{x_n\}$be a sequence in$X$ and $x \in X$.

(a) The sequence $\{x_n\}$ is said to be complex valued convergent in (X,d) and converges to x if for every $\epsilon \succ 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec \epsilon$ for all $n > n_0$ and is denoted by $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is called complex valued Cauchy sequence in (X,d) if $\lim_{x\to\infty} d(x_n, x_{n+p}) = 0$ for all $p \succ 0$.

(c) (X,d) is said to be a complex valued complete rectangular b-metric space if every complex valued Cauchy sequence in X converges to some $x \in X$.

Since the following two lemmas are the analogues of the lemmas in [5], we state these for complex valued rectangular b-metric spaces without their proofs.

Lemma 2.9. *[21] Let (X,d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 2.10. *[21]Let (X,d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Theorem 2.11. *[21]Let* (X, d) *be a complex valued complete rectangular b-metric space with coefficient* $s > 1$ *and* $T : X \to X$ *be a mapping satisfying:*

$$d(Tx, Ty) \preceq \alpha d(x, y)$$

for all $x, y \in X$*, where* $\alpha \in [0, \frac{1}{s})$*. Then* T *has a unique fixed point.*

Chapter 3

Method of research

1. Find books and publications about complex valued rectangular b-metric space and fixed point theorems.
2. Study and analyze the fixed point theorem.
3. Study and analyze the complex valued rectangular b-metric space.
4. Report the progress of the project to the Research and Development Institute of Kamphaeng Phet Rajabhat University.
5. Give necessary and sufficient conditions to obtain fixed point theorems for a the interpolative approach in complex valued rectangular b-metric space.
6. Prove fixed point theorems for a the interpolative approach in complex valued rectangular b-metric space and give an example to illustrate the significance of such results can be applicable in situations while the aforecited.
7. Write the paper and submit for publication in the international journal. Report the completed project to the Research and Development Institute of Kamphaeng Phet Rajabhat University.

Chapter 4

Main Results

In this chapter, Now we prove our main results.

Theorem 4.1. *Let (X, d) be a complete complex valued rectangular b-metric space with coefficient $s > 1$ and suppose that $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \tag{4.1}$$

for all $x, y \in X$,where $\lambda \in \left[0, \dfrac{2}{s+1}\right]$.Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and each $n \in \mathbb{N}$, we define $z_n = Tz_{n-1}$; $\forall n \in \mathbb{N}$.

We consider,

$$\begin{aligned} d(z_1, z_2) &= d(Tz_0, Tz_1) \\ &\leq \lambda[d(z_0, Tz_0) + d(z_1, Tz_1)] \\ &\leq \lambda[d(z_0, z_1) + d(z_1, z_2)], \end{aligned} \tag{4.2}$$

$$\begin{aligned} \Rightarrow \quad d(z_1, z_2) &\leq \lambda d(z_0, z_1) + \lambda d(z_1, z_2) \\ \Rightarrow \quad d(z_1, z_2) - \lambda d(z_1, z_2) &\leq \lambda d(z_0, z_1) \\ \Rightarrow \quad (1-\lambda) d(z_1, z_2) &\leq \lambda d(z_0, z_1) \\ \Rightarrow \quad d(z_1, z_2) &\leq \frac{\lambda d(z_0, z_1)}{(1-\lambda)}, \end{aligned}$$

$$d(z_1, z_2) \leq \left(\frac{\lambda}{1-\lambda}\right) e_0. \tag{4.3}$$

Thus, $\quad d(z_1, z_2) \leq \left(\dfrac{\lambda}{1-\lambda}\right) e_0$, for some $\left[0, \dfrac{2}{s+1}\right]$, where $e_0 = d(z_0, z_1)$.

Note that

$$\begin{aligned} d(z_2, z_3) &= d(Tz_1, Tz_2) \\ &\leq \lambda[d(z_1, Tz_1) + d(z_2, Tz_2)] \\ &\leq \lambda[d(z_1, z_2) + d(z_2, z_3)], \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Rightarrow \quad d(z_2, z_3) &\leq \lambda d(z_1, z_2) + \lambda d(z_2, z_3) \\ \Rightarrow \quad d(z_2, z_3) - \lambda d(z_2, z_3) &\leq \lambda d(z_1, z_2) \end{aligned}$$

$$\Rightarrow \quad (1-\lambda)d(z_2,z_3) \le \lambda d(z_1,z_2)$$
$$\Rightarrow \quad d(z_2,z_3) \le \frac{\lambda d(z_1,z_2)}{(1-\lambda)},$$

$$d(z_2,z_3) \le \left(\frac{\lambda}{1-\lambda}\right)e_1. \tag{4.5}$$

Thus, $d(z_2,z_3) \le \left(\frac{\lambda}{1-\lambda}\right)e_1$, for some $\lambda \in \left[0, \frac{2}{s+1}\right]$, where $e_1 = d(z_1,z_2)$.
Using (4.3) and (4.5), we have

$$\begin{aligned} d(z_2,z_3) &\le \left(\frac{\lambda}{1-\lambda}\right)d(z_1,z_2), \\ &\le \left(\frac{\lambda}{1-\lambda}\right)\left[\left(\frac{\lambda}{1-\lambda}\right)d(z_0,z_1)\right] \\ d(z_2,z_3) &= \left(\frac{\lambda}{1-\lambda}\right)^2 e_0, \ \ for\ some\ \lambda \in \left[0, \frac{2}{s+1}\right]. \end{aligned} \tag{4.6}$$

$$d(z_n,z_{n+1}) \le \left(\frac{\lambda}{1-\lambda}\right)^n e_0, for\ some \lambda \in \left[0, \frac{2}{s+1}\right], where\ e_0 = d(z_0,z_1). \tag{4.7}$$

Let $e_n = d(z_n,z_{n+1}), \forall n \in \mathbb{N}$. By (4.7) , we have

$$\begin{aligned} d(z_n,z_{n+2}) &= d(Tz_{n-1},Tz_{n+1}) \\ &\le \lambda[d(z_{n-1},Tz_{n-1}) + d(z_{n+1},Tz_{n+1})] \\ &\le \lambda[d(z_{n-1},z_n) + d(z_{n+1},z_{n+2})] \\ &\le \lambda d(z_{n-1},z_n) + \lambda d(z_{n+1},z_{n+2}) \\ &\le \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n-1} d(z_0,z_1) + \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n+1} d(z_0,z_1) \\ &\le \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n-1} e_0 + \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 \\ &\le \left(\frac{\lambda}{1-\lambda}\right)^{n-1}\left[\lambda e_0 + \lambda\left(\frac{\lambda}{1-\lambda}\right)^2 e_0\right]. \\ \therefore d(z_n,z_{n+2}) &\le \left(\frac{\lambda}{1-\lambda}\right)^{n-1} e_0^*\ where\ e_0^* = \left[\lambda e_0 + \lambda\left(\frac{\lambda}{1-\lambda}\right)^2 e_0\right] \\ &for\ some\ \lambda \in \left[0, \frac{2}{s+1}\right]. \end{aligned} \tag{4.8}$$

From (4.7), we get

$$
\begin{aligned}
d(z_n, z_{n+2m+1}) &\le s[d(z_n,z_{n+1}) + d(z_{n+1},z_{n+2}) + d(z_{n+2},z_{n+2m+1})] \\
&\le s[d(z_n,z_{n+1}) + d(z_{n+1},z_{n+2}) + s[d(z_{n+2},z_{n+3}) + d(z_{n+3},z_{n+4}) \\
&\quad + d(z_{n+4},z_{n+2m+1})]] \\
&\le s[(e_n + e_{n+1}) + s[(e_{n+2}+e_{n+3}) + s[(e_{n+4}+e_{n+5}) + \ldots + s[(e_{n+2m-2} \\
&\quad + e_{n+2m-1}) + e_{n+2m}) \\
&\le se_n + se_{n+1} + s^2e_{n+2} + s^2e_{n+3} + s^3e_{n+4} + s^3e_{n+5} + \ldots + s^me_{n+2m-2} + \\
&\quad s^me_{n+2m-1} + s^me_{n+2m} \\
&\le s\left[\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0\right] \\
&\quad + s^3\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0\right] + \ldots + s^m\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-2} e_0 + \right. \\
&\quad \left.\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-1} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+2m} e_0\right. \\
&\le \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \ldots\right. \\
&\quad \left. + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-2} e_0 + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m} e_0\right] + \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right. \\
&\quad \left. + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0 + \ldots + s^m\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-1} e_0\right] \\
&\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-2}\right. \\
&\quad \left. + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m}\right] + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4}\right. \\
&\quad \left. + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-2}\right] \\
&\le s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right] \\
&\quad + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right] \\
&\le \left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^{2} + s^2\left(\frac{\lambda}{1-\lambda}\right)^{4} + \ldots\right]\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] \\
&= \frac{1}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}}\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] \\
&= \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left(1 + \left(\frac{\lambda}{1-\lambda}\right)\right)}{1 - s\left(\frac{\lambda}{1-\lambda}\right)^{2}},
\end{aligned}
$$

$$d(z_n, z_{n+2m+1}) \leq \frac{s\left(\frac{\lambda}{1-\lambda}\right)^n e_0 \left(1+\left(\frac{\lambda}{1-\lambda}\right)\right)}{1-s\left(\frac{\lambda}{1-\lambda}\right)^2}. \tag{4.9}$$

From (4.7), (4.8) we have

$$\begin{aligned}
d(z_n, z_{n+2m}) &\leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + d(z_{n+2}, z_{n+2m})] \\
&\leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + s[d(z_{n+2}, z_{n+3}) + d(z_{n+3}, z_{n+4}) + \\
&\quad d(z_{n+4}, z_{n+2m})]] \\
&\leq s[(e_n+e_{n+1})+s[(e_{n+2}+e_{n+3})+s[(e_{n+4}+e_{n+5})+\ldots+s^{m-1}[(e_{2m-4}+ \\
&\quad e_{2m-3}) + s^{m-1}d(z_{n+2m-2}, z_{n+2m}) \\
&\leq se_n + se_{n+1} + s^2e_{n+2} + s^2e_{n+3} + s^3e_{n+4} + s^3e_{n+5} + \ldots + s^{m-1}e_{2m-4} + \\
&\quad s^{m-1}e_{2m-3} + s^{m-1}d(z_{n+2m-2}, z_{n+2m}) \\
&\leq s\left[\left(\frac{\lambda}{1-\lambda}\right)^n e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0\right] \\
&\quad + s^3\left[\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0\right] + \ldots + s^{m-1}\left[\left(\frac{\lambda}{1-\lambda}\right)^{2m-4} e_0\right. \\
&\quad \left. + \left(\frac{\lambda}{1-\lambda}\right)^{2m-3} e_0\right] + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^* \\
&\leq \left[s\left(\frac{\lambda}{1-\lambda}\right)^n e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+2} e_0 + s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+4} e_0 + \ldots\right. \\
&\quad \left. + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-4} e_0\right] + \left[s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0 + s^2\left(\frac{\lambda}{1-\lambda}\right)^{n+3} e_0 + \right. \\
&\quad \left. s^3\left(\frac{\lambda}{1-\lambda}\right)^{n+5} e_0 + \ldots + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{2m-3} e_0\right] + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^* \\
&\leq s\left(\frac{\lambda}{1-\lambda}\right)^n e_0\left[1+s\left(\frac{\lambda}{1-\lambda}\right)^2 + s^2\left(\frac{\lambda}{1-\lambda}\right)^4 + \ldots + s^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{2m-n-4}\right] \\
&\quad + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^2 + s^2\left(\frac{\lambda}{1-\lambda}\right)^4 + \ldots\right. \\
&\quad \left. + s^{m-2}\left(\frac{\lambda}{1-\lambda}\right)^{2m-n-4}\right] + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^* \\
&\leq s\left(\frac{\lambda}{1-\lambda}\right)^n e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^2 + s^2\left(\frac{\lambda}{1-\lambda}\right)^4 + \ldots\right] \\
&\quad + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\left[1 + s\left(\frac{\lambda}{1-\lambda}\right)^2 + s^2\left(\frac{\lambda}{1-\lambda}\right)^4 + \ldots\right] \\
&\quad + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^* \\
&\leq \left[1+s\left(\frac{\lambda}{1-\lambda}\right)^2 + s^2\left(\frac{\lambda}{1-\lambda}\right)^4 + \ldots\right]\left[s\left(\frac{\lambda}{1-\lambda}\right)^n e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right]
\end{aligned}$$

$$+ s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^*$$

$$= \frac{1}{1-s\left(\frac{\lambda}{1-\lambda}\right)^2}\left[s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0 + s\left(\frac{\lambda}{1-\lambda}\right)^{n+1} e_0\right]$$

$$+ s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^*$$

$$d(z_n, z_{n+2m}) \leq \frac{s\left(\frac{\lambda}{1-\lambda}\right)^{n} e_0\left(1+\left(\frac{\lambda}{1-\lambda}\right)\right)}{1-s\left(\frac{\lambda}{1-\lambda}\right)^2} + s^{m-1}\left(\frac{\lambda}{1-\lambda}\right)^{n+2m-3} e_0^*. \quad (4.10)$$

It follows from (4.9) and (4.10) that

$$\lim_{n\to\infty} d(z_n + z_{n+p}) = 0 \ \ \textit{for all } p > 0. \quad (4.11)$$

Thus $\{z_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$\lim_{n\to\infty} z_n = u. \quad (4.12)$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}
d(u, Tu) &\leq s[d(u, z_n) + d(z_n, z_{n+1}) + d(z_{n+1}, Tu)] \\
&= s[d(u, z_n) + d(z_n, z_{n+1}) + d(Tz_n, Tu)] \\
&\leq s[d(u, z_n) + d(z_n, z_{n+1}) + \lambda[d(z_n, Tz_n) + d(u, Tu)]] \\
&\leq sd(u, z_n) + sd(z_n, z_{n+1}) + s\lambda d(z_n, z_{n+1}) + s\lambda d(u, Tu) \\
&\leq s\lim_{n\to\infty} d(u, z_n) + s\lim_{n\to\infty} d(z_n, z_{n+1}) \\
&\quad + s\lambda \lim_{n\to\infty} d(z_n, z_{n+1}) + s\lambda d(u, Tu) \\
&\leq s\lambda d(u, Tu), \quad \text{by (4.11) and (4.12)}
\end{aligned}$$

Thus $0 \leq d(u, Tu) \leq s\lambda d(u, Tu);\ s\lambda < 1.$

It follows from above inequality that $d(u, Tu) = 0$, i.e. ,$Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then it follows that $0 \leq d(u, v) = d(Tu, Tv) \leq \lambda[d(u, Tu) + d(v, Tv)] = \lambda[d(u, u) + d(v, v)] = 0$ a contradiction. Therefore, we must have $d(u, v) = 0$, i.e.,$u = v$. Thus fixed point is unique. □

Theorem 4.2. *Let (X, d) be a complete complex valued rectangular b-metric space with coefficient $s > 1$ and suppose that $T : X \to X$ be a mapping satisfying:*

$$d(Tx, Ty) \leq k \; max \; \{d(x, y), d(x, Tx), d(y, Ty)\} \tag{4.13}$$

for all $x, y \in X$, where $k \in \left[0, \frac{2}{s+1}\right]$.Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and each $n \in \mathbb{N}$, we define $z_n = Tz_{n-1}$; $\forall n \in \mathbb{N}$.

We consider,

$$\begin{aligned} d(z_1, z_2) &= d(Tz_0, Tz_1) \\ &\leq k \; max \; \{d(z_0, z_1), d(z_0, Tz_0), d(z_1, Tz_1)\} \\ &\leq k \; max \; \{d(z_0, z_1), d(z_0, z_1), d(z_1, z_2)\} \\ &= k \; max \; \{d(z_0, z_1), d(z_1, z_2)\} \\ &\leq k[d(z_0, z_1) + d(z_1, z_2)], \\ d(z_1, z_2) &\leq kd(z_0, z_1) + kd(z_1, z_2). \end{aligned} \tag{4.14}$$

$$\Rightarrow \; d(z_1, z_2) \leq \left(\frac{k}{1-k}\right) d(z_0, z_1). \tag{4.15}$$

Thus, $d(z_1, z_2) \leq \left(\frac{k}{1-k}\right) e_0$, for some $k \in \left[0, \frac{2}{s+1}\right]$, where $e_0 = d(z_0, z_1)$.

Note that

$$\begin{aligned} d(z_2, z_3) &= d(Tz_1, Tz_2) \\ &\leq k \; max \; \{d(z_1, z_2), d(z_1, Tz_1), d(z_2, Tz_2)\} \\ &\leq k \; max \; \{d(z_1, z_2), d(z_1, z_2), d(z_2, z_3)\} \\ &= k \; max \; \{d(z_1, z_2), d(z_2, z_3)\} \\ &\leq k[d(z_1, z_2) + d(z_2, z_3)], \\ d(z_2, z_3) &\leq kd(z_1, z_2) + kd(z_2, z_3). \end{aligned} \tag{4.16}$$

$$\Rightarrow \; d(z_2, z_3) \leq \left(\frac{k}{1-k}\right) d(z_1, z_2). \tag{4.17}$$

Thus, $d(z_2, z_3) \leq \left(\frac{k}{1-k}\right) e_1$, for some $k \in \left[0, \frac{2}{s+1}\right]$, where $e_1 = d(z_1, z_2)$.

Using (4.15) and (4.17), we have

$$\begin{aligned}
d(z_2, z_3) &\leq \left(\frac{k}{1-k}\right) d(z_1, z_2),\\
&\leq \left(\frac{k}{1-k}\right)\left[\left(\frac{k}{1-k}\right) d(z_0, z_1)\right]\\
d(z_2, z_3) &= \left(\frac{k}{1-k}\right)^2 e_0,\ \ for\ some\ k \in \left[0, \frac{2}{s+1}\right]. \qquad (4.18)
\end{aligned}$$

$$d(z_n, z_{n+1}) \leq \left(\frac{k}{1-k}\right)^n e_0, for\ some\ k \in \left[0, \frac{2}{s+1}\right], where\ e_0 = d(z_0, z_1) \qquad (4.19)$$

Let $e_n = d(z_n, z_{n+1}), \forall n \in \mathbb{N}$. By (4.19), we have

$$\begin{aligned}
d(z_n, z_{n+2}) &= d(Tz_{n-1}, Tz_{n+1})\\
&\leq k\ max\{d(z_{n-1}, z_{n+1}), d(z_{n-1}, Tz_{n-1}), d(z_{n+1}, Tz_{n+1})\}\\
&\leq k\ max\{d(z_{n-1}, z_{n+1}), d(z_{n-1}, z_n), d(z_{n+1}, z_{n+2})\}\\
&\leq k\ max\{s[d(z_{n-1}, z_n) + d(z_n, z_{n+2}) + d(z_{n+2}, z_{n+1})], d(z_{n-1}, z_n), d(z_{n+1}, z_{n+2})\}\\
&\leq k\ max\{sd(z_{n-1}, z_n) + sd(z_n, z_{n+2}) + sd(z_{n+2}, z_{n+1}), d(z_{n-1}, z_n), d(z_{n+1}, z_{n+2})\}\\
&\leq k[sd(z_{n-1}, z_n) + sd(z_n, z_{n+2}) + sd(z_{n+2}, z_{n+1})]\\
d(z_n, z_{n+2}) &\leq ksd(z_{n-1}, z_n) + ksd(z_n, z_{n+2}) + ksd(z_{n+2}, z_{n+1}) \qquad (4.20)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & d(z_n, z_{n+2}) \leq ksd(z_{n-1}, z_n) + ksd(z_n, z_{n+2}) + ksd(z_{n+2}, z_{n+1})\\
\Rightarrow\quad & d(z_n, z_{n+2}) - ksd(z_n, z_{n+2}) \leq ksd(z_{n-1}, z_n) + ksd(z_{n+2}, z_{n+1})\\
\Rightarrow\quad & [1-ks]d(z_n, z_{n+2}) \leq ks[d(z_{n-1}, z_n) + d(z_{n+2}, z_{n+1})]\\
\Rightarrow\quad & d(z_n, z_{n+2}) \leq \left(\frac{ks}{1-ks}\right)[d(z_{n-1}, z_n) + d(z_{n+1}, z_{n+2})]\\
\Rightarrow\quad & d(z_n, z_{n+2}) \leq \left(\frac{ks}{1-ks}\right)\left[\left(\frac{k}{1-k}\right)^{n-1} e_0 + \left(\frac{k}{1-k}\right)^{n+1} e_0\right]\\
\Rightarrow\quad & d(z_n, z_{n+2}) \leq \left(\frac{k}{1-k}\right)^{n-1}\left[\left(\frac{ks}{1-ks}\right) e_0 + \left(\frac{ks}{1-ks}\right)\left(\frac{k}{1-k}\right)^2 e_0\right]\\
\Rightarrow\quad & d(z_n, z_{n+2}) \leq \left(\frac{k}{1-k}\right)^{n-1} e_0^*
\end{aligned}$$

$$e_0^* = \left[\left(\frac{ks}{1-ks}\right) e_0 + \left(\frac{ks}{1-ks}\right)\left(\frac{k}{1-k}\right)^2 e_0\right] for\ some\ k \in \left[0, \frac{2}{s+1}\right] \qquad (4.21)$$

From (4.19), we get

$$d(z_n, z_{n+2m+1}) \leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + d(z_{n+2}, z_{n+2m+1})]$$

$$
\begin{aligned}
&\leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + s[d(z_{n+2}, z_{n+3}) + d(z_{n+3}, z_{n+4}) + \\
&\quad d(z_{n+4}, z_{n+2m+1})]] \\
&\leq s[(e_n+e_{n+1})+s[(e_{n+2}+e_{n+3})+s[(e_{n+4}+e_{n+5})+...+s[(e_{n+2m-2}+ \\
&\quad e_{n+2m-1}) + e_{n+2m}) \\
&\leq se_n+se_{n+1}+s^2e_{n+2}+s^2e_{n+3}+s^3e_{n+4}+s^3e_{n+5}+...+s^me_{n+2m-2}+ \\
&\quad s^me_{n+2m-1} + s^me_{n+2m} \\
&\leq s\left[\left(\frac{k}{1-k}\right)^{n} e_0 + \left(\frac{k}{1-k}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{k}{1-k}\right)^{n+2} e_0 + \right. \\
&\quad \left.\left(\frac{k}{1-k}\right)^{n+3} e_0\right] + s^3\left[\left(\frac{k}{1-k}\right)^{n+4} e_0 + \left(\frac{k}{1-k}\right)^{n+5} e_0\right] + \\
&\quad ...+s^m\left[\left(\frac{k}{1-k}\right)^{n+2m-2} e_0 + \left(\frac{k}{1-k}\right)^{n+2m-1} e_0 + \left(\frac{k}{1-k}\right)^{n+2m} e_0\right] \\
&\leq \left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+2} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+4} e_0 + ...\right. \\
&\quad \left. + s^m\left(\frac{k}{1-k}\right)^{n+2m-2} e_0 + s^m\left(\frac{k}{1-k}\right)^{n+2m} e_0\right] + \left[s\left(\frac{k}{1-k}\right)^{n+1} e_0 + \right. \\
&\quad \left. s^2\left(\frac{k}{1-k}\right)^{n+3} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+5} e_0 + ... + s^m\left(\frac{k}{1-k}\right)^{n+2m-1} e_0\right] \\
&\leq s\left(\frac{k}{1-k}\right)^{n} e_0 \left[1+s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + ... + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-2}\right. \\
&\quad \left. + s^{m-1}\left(\frac{k}{1-k}\right)^{2m}\right] + s\left(\frac{k}{1-k}\right)^{n+1} e_0 \left[1+s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4}\right. \\
&\quad \left. + ... + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-2}\right] \\
&\leq s\left(\frac{k}{1-k}\right)^{n} e_0 \left[1+s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + ...\right] + \\
&\quad s\left(\frac{k}{1-k}\right)^{n+1} e_0 \left[1+s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + ...\right] \\
&\leq \left[1+s\left(\frac{k}{1-k}\right)^{2} + s^2\left(\frac{k}{1-k}\right)^{4} + ...\right]\left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0\right] \\
&= \frac{1}{1-s\left(\frac{k}{1-k}\right)^{2}}\left[s\left(\frac{k}{1-k}\right)^{n} e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0\right] \\
&= \frac{s\left(\frac{k}{1-k}\right)^{n} e_0 \left(1+\left(\frac{k}{1-k}\right)\right)}{1-s\left(\frac{k}{1-k}\right)^{2}},
\end{aligned}
$$

$$d(z_n, z_{n+2m+1}) \leq \frac{s\left(\frac{k}{1-k}\right)^n e_0\left(1+\left(\frac{k}{1-k}\right)\right)}{1-s\left(\frac{k}{1-k}\right)^2}. \tag{4.22}$$

From (4.9), (4.21) we have

$$\begin{aligned}
d(z_n, z_{n+2m}) &\leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + d(z_{n+2}, z_{n+2m})] \\
&\leq s[d(z_n, z_{n+1}) + d(z_{n+1}, z_{n+2}) + s[d(z_{n+2}, z_{n+3}) + d(z_{n+3}, z_{n+4}) \\
&\quad + d(z_{n+4}, z_{n+2m})]] \\
&\leq s[(e_n + e_{n+1}) + s[(e_{n+2} + e_{n+3}) + s[(e_{n+4} + e_{n+5}) + \ldots \\
&\quad + s^{m-1}[(e_{2m-4} + e_{2m-3}) + s^{m-1} d(z_{n+2m-2}, z_{n+2m}) \\
&\leq se_n + se_{n+1} + s^2 e_{n+2} + s^2 e_{n+3} + s^3 e_{n+4} + s^3 e_{n+5} + \ldots \\
&\quad + s^{m-1} e_{2m-4} + s^{m-1} e_{2m-3} + s^{m-1} d(z_{n+2m-2}, z_{n+2m}) \\
&\leq s\left[\left(\frac{k}{1-k}\right)^n e_0 + \left(\frac{k}{1-k}\right)^{n+1} e_0\right] + s^2\left[\left(\frac{k}{1-k}\right)^{n+2} e_0 + \left(\frac{k}{1-k}\right)^{n+3} e_0\right] + \\
&\quad s^3\left[\left(\frac{k}{1-k}\right)^{n+4} e_0 + \left(\frac{k}{1-k}\right)^{n+5} e_0\right] + \ldots + s^{m-1}\left[\left(\frac{k}{1-k}\right)^{2m-4} e_0 + \right. \\
&\quad \left. s\left(\frac{k}{1-k}\right)^{n+1} s\left(\frac{k}{1-k}\right)^{n+1}\left(\frac{k}{1-k}\right)^{2m-3} e_0\right] + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^* \\
&\leq \left[s\left(\frac{k}{1-k}\right)^n e_0 + s^2\left(\frac{k}{1-k}\right)^{n+2} e_0 + s^3\left(\frac{k}{1-k}\right)^{n+4} e_0 + \ldots\right. \\
&\quad \left. + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-4} e_0\right] + \left[s\left(\frac{k}{1-k}\right)^{n+1} e_0 + s^2\left(\frac{k}{1-k}\right)^{n+3} e_0\right. \\
&\quad \left. + s^3\left(\frac{k}{1-k}\right)^{n+5} e_0 + \ldots + s^{m-1}\left(\frac{k}{1-k}\right)^{2m-3} e_0\right] + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^* \\
&\leq s\left(\frac{k}{1-k}\right)^n e_0\left[1 + s\left(\frac{k}{1-k}\right)^2 + s^2\left(\frac{k}{1-k}\right)^4 + \right. \\
&\quad \left. \ldots + s^{m-2}\left(\frac{k}{1-k}\right)^{2m-n-4}\right] + s\left(\frac{k}{1-k}\right)^{n+1} e_0\left[1 + s\left(\frac{k}{1-k}\right)^2\right. \\
&\quad \left. + s^2\left(\frac{k}{1-k}\right)^4 \ldots + s^{m-2}\left(\frac{k}{1-k}\right)^{2m-n-4}\right] + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^* \\
&\leq s\left(\frac{k}{1-k}\right)^n e_0\left[1 + s\left(\frac{k}{1-k}\right)^2 + s^2\left(\frac{k}{1-k}\right)^4 + \ldots\right] + \\
&\quad s\left(\frac{k}{1-k}\right)^{n+1} e_0\left[1 + s\left(\frac{k}{1-k}\right)^2 + s^2\left(\frac{k}{1-k}\right)^4 + \ldots\right] \\
&\quad + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^* \\
&\leq \left[1 + s\left(\frac{k}{1-k}\right)^2 + s^2\left(\frac{k}{1-k}\right)^4 + \ldots\right]\left[s\left(\frac{k}{1-k}\right)^n e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0\right]
\end{aligned}$$

$$+ s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^*$$

$$= \frac{1}{1-s\left(\frac{k}{1-k}\right)^2}\left[s\left(\frac{k}{1-k}\right)^n e_0 + s\left(\frac{k}{1-k}\right)^{n+1} e_0\right]$$

$$+ s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^*$$

$$d(z_n, z_{n+2m}) \leq \frac{s\left(\frac{k}{1-k}\right)^n e_0 \left(1+\left(\frac{k}{1-k}\right)\right)}{1-s\left(\frac{k}{1-k}\right)^2} + s^{m-1}\left(\frac{k}{1-k}\right)^{n+2m-3} e_0^*. \quad (4.23)$$

It follows from (4.22) and (4.23) that

$$\lim_{n\to\infty} d(z_n + z_{n+p}) = 0 \ \ for\ all\ p > 0. \quad (4.24)$$

Thus $\{z_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ such that

$$\lim_{n\to\infty} z_n = u. \quad (4.25)$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$

we have

$$\begin{aligned}
d(u, Tu) &\leq s[d(u, z_n) + d(z_n, z_{n+1}) + d(z_{n+1}, Tu)] \\
&= s[d(u, z_n) + d(z_n, z_{n+1}) + d(Tz_n, Tu)] \\
&\leq s[d(u, z_n) + d(z_n, z_{n+1}) + k\, max\{d(z_n, u), d(z_n, Tz_n), d(u, Tu)\}] \\
&\leq s[d(u, z_n) + d(z_n, z_{n+1}) + k\, max\{d(z_n, u), d(z_n, z_{n+1}), d(u, Tu)\}] \\
&\leq s[d(u, z_n) + d(z_n, z_{n+1}) + k\{d(z_n, u), d(z_n, z_{n+1}), d(u, Tu)\}] \\
&\leq sd(u, z_n) + sd(z_n, z_{n+1}) + skd(z_n, u) + skd(z_n, z_{n+1}) + skd(u, Tu) \\
&\leq s \lim_{n\to\infty} d(u, z_n) + s \lim_{n\to\infty} d(z_n, z_{n+1}) + sk \lim_{n\to\infty} d(z_n, u) \\
&\quad + sk \lim_{n\to\infty} d(z_n, z_{n+1}) + skd(u, Tu),
\end{aligned}$$

$d(u, Tu) \leq skd(u, Tu)$. by (4.24) and (4.25)

Thus $\quad 0 \leq d(u, Tu) \leq skd(u, Tu);\ sk < 1$.

It follows from above inequality that $d(u, Tu) = 0$, i.e. ,$Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then it follows that $d(u, v) = d(Tu, Tv) \leq k\, max\{d(u, v), d(u, Tu), d(v, Tv)\} \leq kd(u, v)$. Thus a contradiction. Therefore, we must have $d(u, v) = 0$, i.e.,$u = v$. Thus fixed point is unique. □

Chapter 5

Conclusion

The purpose of this paper is to study the existence of fixed point for self mappings under generalized kannan mappings type concept in b-metric spaces. Our result extend and generalize the result derived by Ege [21] and many others.Moreover, we give examples as a satisfying the theorems in complex valued rectangular b-metric spaces as follows:

Theorem 1 Let (X, d) be a complete complex valued rectangular b-metric space with coefficient $s > 1$ and suppose that $T : X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$,where $\lambda \in \left[0, \frac{2}{s+1}\right]$. Then T has a unique fixed point.

Theorem 2 Let (X, d) be a complete complex valued rectangular b-metric space with coefficient $s > 1$ and suppose that $T : X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \leq k\ max\,\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$,where $k \in \left[0, \frac{2}{s+1}\right]$. Then T has a unique fixed point.

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Appendix


การประชุมวิชาการระดับชาติครั้ง ที่ 6
The 6th KPRU National Conference